MA 222 - Analysis II: Measure and Integration (JAN-APR, 2016)

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Problem set 1 (Preliminaries)

1. Sketch the following functions:
   (a) $x,\ x^2,\ x^3,\ |x|,\ x|x|,\ \sin x,\ \exp(x),\ \exp(-x)$ in $\mathbb{R}$
   (b) $\log x,\ \sin(\frac{1}{x}),\ x\sin(\frac{1}{x})$ in $(0, \infty)$

2. Analyze the continuity and differentiability of the above functions. Construct a function which is n-times differentiable nut not $n+1$ times.

3. Let $f(x) = \dfrac{1}{x^\alpha}$ in $(0,1)$ where $0 < \alpha \leq 1$. Is f continuous? What about uniform continuity? justify.

4. Suppose f is a real function defined on $\mathbb{R}$ that satisfies $\lim_{h\to 0}(f(x+h) - f(x-h)) = 0,\ \forall x \in \mathbb{R}$. Does this imply that f is continuous?

5. Check right and left discontinuities of the largest integer function $[x]$ and fractional function $\{x\} = x - [x]$.

6. a) Find the upper and lower limits of the sequence given by $x_1 = 0,\ x_{2m} = \dfrac{1}{2}x_{2m-1},\ x_{2m+1} = 1 + x_{2m}$.
   b) Prove that the convergence of x_n implies the convergence of $|x_n|$. What about converse?

7. Prove that any countable union of open sets is open and countable intersection of closed sets is closed. What about countable intersection of open sets (*known as G_δ sets*) and countable union of closed sets (*known as F_σ sets*).

8. Show that $\underline{\lim}\ x_n \leq \overline{\lim}\ x_n$ (Recall $\underline{x} = \underline{\lim}\ x_n = \sup_n \inf_{k\geq n} x_k$ and $\overline{x} = \overline{\lim}\ x_n = \inf_n \sup_{k\geq n} x_k$). Further prove, $l = \underline{\lim}\ x_n = \overline{\lim}\ x_n$ if and only if $l = \lim\ x_n$.

9. If $\underline{x} = \lim\ x_n$ finite if and only if (i) given $\epsilon > 0,\ \exists\, N$ such that $x_k \geq \underline{x} - \epsilon, \forall\ k \geq N$ and (ii) given $\epsilon > 0,\ \forall\ n,\ \exists\ k \geq n$ such that $x_k \leq \underline{x} + \epsilon$.

10. Let E be the set of cluster points of $\{x_k\}$. Then, show that $\overline{x} = \sup E$ and $\underline{x} = \inf E$. (l is called a cluster point of x_k if every neighborhood of l contains a point from $\{x_k\}$).

11. Give lower and upper limits of sequence of sets.

12. Consider $f_n(x) = x^n,\ 0 \leq x \leq 1$. Let $f(x) = \lim f_n(x)$. Find f. Is the convergence uniform in [0,1]? Is the convergence uniform in [0,δ], where $0 < \delta < 1$?

13. Consider

$$f_n(x) = \begin{cases} n^2 x\ , & 0 \leq x \leq \frac{1}{n} \\ -n^2 x + 2n\ , & \frac{1}{n} \leq x \leq \frac{n}{n} \\ 0\ , & \frac{2}{n} \leq x \leq 1 \end{cases}$$

Find $f(x) = \lim_{n\to\infty} f_n(x)$. Are $f_n,\ f$ continuous? Is the convergence uniform? Find $\int_0^1 f_n(x)$ and $\int_0^1 f(x)$. What about $\int_0^1 f_n(x) \to \int_0^1 f(x)$?

14. Show, if $f_n \to f$ uniformly in [0.1] and $f_n,\ f$ are Riemann integrable, then $\int_0^1 f_n(x) \to \int_0^1 f(x)$.

15. Let $A = \mathbb{Q} \cap [0,1],\ B = [0,1] \backslash A$. Arrange A as A=$\{x_1, x_2, \cdots\}$.
Define $f_n(x) = \begin{cases} 1\ ,\ if & x \in \{x_1, x_2, \cdots x_n\} \\ 0\ , & \text{otherwise} \end{cases}$
Is f_n Riemann integrable? Find $f = \lim_{n\to\infty} f_n(x)$. Is f Riemann integrable?

16. Give an example of a function defined on [0,1] which is Riemann Integrable but has infinitely many discontinuities.

17. a) Let A=[0,1] $\cap\mathbb{Q}$. Given any $\epsilon > 0$, construct a countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ such that $A \subset \bigcup_{k=1}^{\infty} I_k$ and $\Sigma_{k=1}^{\infty} l(I_k) \leq \epsilon$.
b) On the other hand if $\{I_k\}_{k=1}^{n}$ is a finite open cover of A, then show that $\Sigma_{k=1}^{n} l(I_k) \geq 1$

18. Let $f : [0,1] \to \mathbb{R}$ be continuous and $\int_0^1 |f(x)|dx = 0$. Show that $f(x) = 0,\ \forall\ x \in [0,1]$.

19. Let $f : [0,1] \to \mathbb{R}$ be continuous and $\int_0^y f(x)dx = 0\ \forall\ y \in [0,1]$. Show that $f \equiv 0\ in\ [0,1]$.